On the numerical solution of the flow between a rotating and a stationary disk

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ABSTRACT

The flow between two co-axial, infinite disks, one rotating with constant angular velocity and one stationary is treated in this paper. The problem is reduced to that of finding the solution of a two-point boundary value for a sixth order nonlinear ordinary differential equation and three boundary conditions at each end of a finite interval. The numerical solutions are obtained by using a fourth order Runge-Kutta integration scheme in modification due to Gill and in conjunction with a modified shooting method to correct the initial guesses at one boundary. The numerical calculations for different Reynolds numbers are carried out. The results obtained by this method are compared with available results. The comparison shows excellent agreement.

INTRODUCTION

The steady state viscous incompressible flow between two infinite disks, one rotating and the other stationary, has been discussed by many early workers. By assuming the axial velocity to be radius independent, Karman [1] obtained a set of ordinary differential equations from the full Navier-Stokes equations which describe the steady flow of a viscous incompressible fluid over an infinitely large rotating disk. This problem is of considerable interest because of the possibility of obtaining exact solutions to the full Navier-Stokes equations for any Reynolds numbers. Batchelor [2] has generalized the Karman method to the case of two rotating disks, and has discussed the nature of the steady flow between the two disks. Further comments have been made by Stewartson [3], who argued that the free disk solution of Karman was in fact the proper limiting case for large Reynolds numbers.
The same problem has been discussed by Lance and Rogers [4]. They have obtained the numerical solution by a shooting method. Pearson [5] has generalized it for the impulsively started and counter rotating disks. More recently, Mellor et al. [6] investigated the case where one disk rotated and the other remained stationary, and demonstrated that the Karman solution is the limit solution of a certain branch of two disks solution as the Reynolds number increases. Furthermore, they showed that many other solutions for a single Reynolds number are possible. To obtain the solution, they transformed the problem to an initial value one and then they employed the third order Runge-Kutta integration scheme.

The scope of the present paper is to demonstrate an elegant numerical procedure to obtain the solution of the problem described by Mellor et al. [6]. The method consists in employing the fourth order Runge-Kutta integration scheme in the modification due to Gill (see Ralston [7]) to a set of simultaneous first order ordinary differential equations which govern the flow phenomenon in conjunction with a modified shooting method to correct the initial guesses at one boundary. The merit of this method rests on the fact that it is self starting and the assumed values at one end are changed automatically in a systematic way until the correct starting values have been determined.

1. EQUATIONS OF MOTION

The governing equations for an incompressible flow are its continuity and the three equations of motion. Cylindrical polar coordinates (r, θ, z) are used and the fluid fills the space between the disks which lie in the plane $z = 0$, the stationary disk position and $z = L$, the rotating disk position and Ω is its angular velocity. Following Mellor et al. [6], we look for axially symmetric solutions of the Navier-Stokes equations in the form :

$$\begin{aligned} u &= r\Omega\, f'(\eta) \\ v &= r\Omega\, h(\eta) \\ w &= -2\Omega L\, f(\eta) \\ (P/\rho) &= \Omega^2 L^2 P_0(\eta) + \frac{1}{2}\lambda\, \Omega^2 r^2 \\ z &= L\eta \end{aligned} \tag{1.1}$$

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where λ is an absolute constant and u, v, w are respectively the radial, transverse and axial components of the fluid velocity. When substituted into the full Navier-Stokes equations, the ordinary nonlinear equations which the functions f, f′ and h must satisfy are :

$$
\begin{aligned}
f''' &= R(\lambda - h^2 - 2f\,f'' + f'^2) \\
h'' &= -2R\,(fh' - f'h) \qquad (1.2)\\
P_0' &= -2\,(2f\,f' + \frac{f''}{R})
\end{aligned}
$$

where $R = (\Omega L^2/\nu)$ the Reynolds number, ν is the kinematic viscosity of the fluid and a prime denotes differentiation with respect to η.

The relevant boundary conditions are the following :

$$f(0) = f'(0) = 0,\ h(0) = 0,\ \text{at } \eta = 0, \qquad (1.3)$$

$$f(1) = f'(1) = 0,\ h(1) = 1,\ \text{at } \eta = 1. \qquad (1.4)$$

To get rid of λ, the arbitrary constant, we differentiate the first of the equations (1.2) and see that the equations (1.2) submit to be sixth order nonlinear ordinary differential equations subject to six boundary conditions. These equations are :

$$f^{iv} = -2R\,(f\,f''' + h\,h') \qquad (1.5)$$

$$h'' = -2R\,(f\,h' - f'h) \qquad (1.6)$$

$$P_0' = -2\,(2\,f\,f' + \frac{f''}{R}) \qquad (1.7)$$

It is obvious that the pressure function $P_0(\eta)$ can be obtained immediately after the simultaneous equations in f and h function in (1.5) and (1.6) are determined. Thus the problem is reduced to determining the solutions to the two equations (1.5) and (1.6). These equations are valid for finite Reynolds numbers.

2. NUMERICAL METHOD

In this section we describe the method for numerical computation. Equations (1.5) and (1.6) are sixth order nonlinear ordinary differential equations. Thus we must have six initial conditions, i.e. six conditions at one end are needed for numerical integration of the system.

At $\eta = 0$, three conditions are given by the definition of the problem; let us assume that the other three missing boundary conditions at $\eta = 0$ are :

$$
\begin{aligned}
f''\ (0) &= A \\
f'''\ (0) &= B \qquad (2.1)\\
h'\ (0) &= C
\end{aligned}
$$

where A, B and C are the initial guesses. The problem now is reduced to finding these values for which the other conditions at $\eta = 1$, i.e. $f(1) = f'(1) = 0$ and $h(1) = 1$ are satisfied. Thus it is seen that the solution system is a function of A, B and C.

$$
\begin{aligned}
f &= f\,(\eta, A, B, C) \\
f' &= f'(\eta, A, B, C) \qquad (2.2)\\
h &= h\,(\eta, A, B, C)
\end{aligned}
$$

Suppose $A + k_1$, $B + k_2$ and $C + k_3$ are the correct initial guesses for which the conditions at $\eta = 1$ are satisfied; then we have :

$$
\begin{aligned}
f\ (1, A + k_1,\ B + k_2,\ C + k_3) &= 0 \\
f'\ (1, A + k_1,\ B + k_2,\ C + k_3) &= 0 \qquad (2.3)\\
h\ (1, A + k_1,\ B + k_2,\ C + k_3) &= 1
\end{aligned}
$$

Now if k_1, k_2, k_3 are small such that $0(k_1^2, k_2^2, k_3^2)$ can be neglected, then expanding them by Taylor series,

$$
\begin{aligned}
f\ (1, A, B, C) + k_1 f_A + k_2 f_B + k_3 f_C &= 0 \\
f'\ (1, A, B, C) + k_1 f_A' + k_2 f_B' + k_3 f_C' &= 0 \qquad (2.4)\\
h\ (1, A, B, C) + k_1 h_A + k_2 h_B + k_3 h_C &= 1
\end{aligned}
$$

where a subscript means the partial differentiation with respect to the appropriate parameter.

Evaluating the unknown constants k_1, k_2 and k_3 we obtain,

$$\Delta k_1 = \begin{vmatrix} -f & f_B & f_C \\ -f' & f_B' & f_C' \\ 1-h & h_B & h_C \end{vmatrix} \qquad (2.5)$$

$$\Delta k_2 = \begin{vmatrix} f_A & -f & f_C \\ f_A' & -f' & f_C' \\ h_A & 1-h & h_C \end{vmatrix} \qquad (2.6)$$

$$\Delta k_3 = \begin{vmatrix} f_A & f_B & -f \\ f_A' & f_B' & -f' \\ h_A & h_B & 1-h \end{vmatrix} \qquad (2.7)$$

$$\Delta = \begin{vmatrix} f_A & f_B & f_C \\ f_A' & f_B' & f_C' \\ h_A & h_B & h_C \end{vmatrix} \qquad (2.8)$$

Note that all entries of the determinant must be evaluated at $\eta = 1$. The partial derivatives that appear in solving the quantities k_1, k_2 and k_3 are to be obtained from the numerical integration of the following differential equations :

$$f_n^{iv} = -2R(f_n f''' + f f_n''' + h_n h' + h_n' h) \qquad (2.9)$$

$$h_n'' = -2R(f_n h' + f h_n' - f_n' h - f' h_n) \qquad (2.10)$$

where a subscript denotes the partial differentiation with respect to n and n takes the values A, B, and C. Thus with this arrangement, the boundary value problem goes over to the initial value one. The initial conditions for the system of equations (1.5), (1.6), (2.9) and (2.10) are the following :

$$
\begin{array}{ll}
f\ (0) = 0 & f_n(0) = 0 \\
f'\ (0) = 0 & f'_n(0) = 0 \\
f''\ (0) = A & f''_n(0) = \delta^{nA} \\
f'''(0) = B & f'''_n(0) = \delta^{nB} \\
h\ (0) = 0 & h_n(0) = 0 \qquad n = A, B, C. \\
h'\ (0) = C & h'_n(0) = \delta^{nC}
\end{array} \tag{2.11}
$$

where δ^{ij} is a Kronecker delta such that

$$
\begin{aligned}
\delta^{ij} &= 0 \quad \text{if} \quad i \neq j \\
&= 1 \quad \text{if} \quad i = j\,.
\end{aligned}
$$

The differential equations (1.5), (1.6), (2.9) and (2.10) are written in the form of a system of first order differential equations. In order to integrate the system from $\eta = 0$ to $\eta = 1$, it is necessary to assume values for A, B and C. A fourth order Runge-Kutta integration scheme in the modification due to Gill is employed to obtain in the values of k_1, k_2 and k_3.
The new values of A, B and C are computed by the following algorithm :

$$
\begin{aligned}
A^{m+1} &= A^m + k_1^m \\
B^{m+1} &= B^m + k_2^m \\
C^{m+1} &= C^m + k_3^m
\end{aligned} \tag{2.12}
$$

where A^{m+1} is the $(m+1)$th approximation to A. Similarly for B^{m+1} and C^{m+1}.
This iteration procedure should be carried out until the square root of the function
$f^2(1) + f'^2(1) + [1 - h(1)]^2$ is less than ϵ, where ϵ is a preassigned value as small as possible. If this condition is satisfied this means that the functions f, f′ and h have satisfied the boundary conditions at $\eta = 1$. At this stage all the boundary conditions are satisfied and the solution can be printed out.

3. NUMERICAL RESULTS AND CONCLUSION

Numerical results obtained by this method are displayed in figures 1 and 2 for Reynolds numbers R up to 107. The integration and iteration procedure outlined in the preceding section were applied until the square root of the function $f^2(1) + f'^2(1) + [1 - h(1)]^2$ is less than ϵ, where $\epsilon = 10^{-4}$. This condition guarantees that the boundary conditions at $\eta = 1$ are satisfied by the functions f, f′ and h with an error of $0(10^{-4})$. The computations were carried out by starting with $R = 1$ with the guessed values $A = 1$, $B = 1$ and $C = 1$. The solutions for this case were obtained after two iterations and within a fraction of a second of computer time up to the accuracy mentioned above. Thus with the knowledge of these values, the computations were performed for $R = 5$, and successively, for high Reynolds numbers, solutions were obtained with the knowledge of the preceeding values. In the course of calculation we found that two to six iterations were needed to obtain a solution and iteration could be made to converge up to Reynolds number 107, but beyond that it was hard to bring about convergence. This difficulty was also reported by Mellor et al. [6]. We also found that for large values of R, small changes in A, B and C produce large changes in the end values f(1), f′(1) and h(1), and therefore this method cannot be used further. Lance and Rogers [4] have adopted a method due to Fox [8] to obtain solutions for large Reynolds numbers, which essentially consists of starting at both ends of the range of integration and matching these two solutions at an intermediate point. But in our computation we have not made any use of this method to obtain the solution for large Reynolds numbers beyond $R = 107$. Use of this method will be rather redundant and duplication of the previous work. The numerical results are presented in graphical form. Figure 1 shows the solutions of f and h for Reynolds numbers $R = 5$, 10, 50 and 107. Figure 2 displays the results of f′ for Reynolds numbers $R = 5$, 10, 50 and 107. Table 1 shows the comparison of our results with those of Mellor et al. [6].
The method reported here is powerful for the Reynolds numbers range indicated in the sense that it is self-starting and the guessed values are corrected automatically by the routine itself. The maximum iteration needed to obtain a solution is up to six and the computation time varies from a fraction of a second to about four seconds depending upon Reynolds numbers.

ACKNOWLEDGEMENT

The financial support from the National Research Council of Canada is gratefully acknowledged. The author is also thankful to J. Ploeg, Head of the Hydraulics Laboratory, for his constant encouragement.

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Table 1

Comparison of our results with those of Mellor et al. [6]

	Mellor et al. [6]	Author
R	$H'''(0)$	$f'''(0)$
3.1	0.90830	0.907931
9.7	2.25040	2.364198
19.2	3.12960	3.132681
30.1	3.13040	3.129698
46.5	2.85045	2.852480
67.9	2.79748	2.786717
96.9	3.77910	3.827744

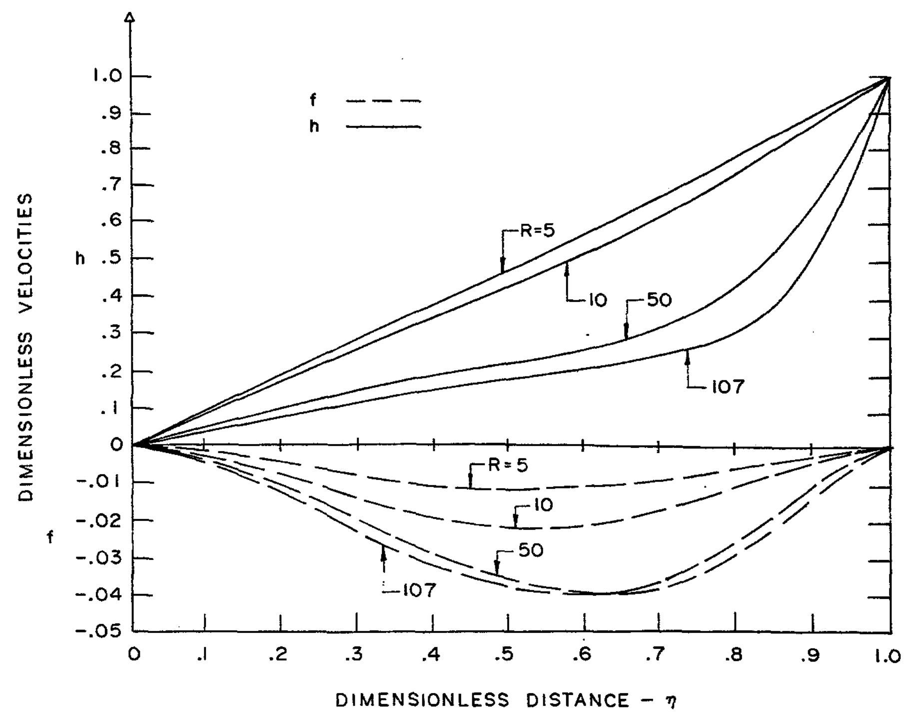

Fig. 1. Numerical results of f and h for R = 5, 10, 50 and 107.

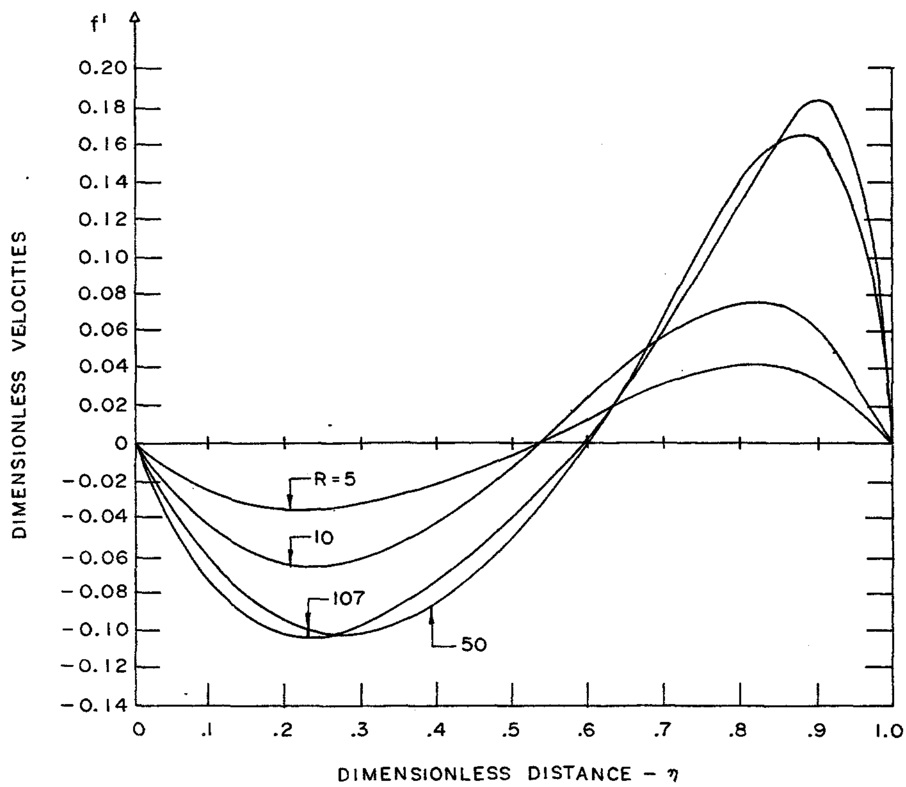

Fig. 2. Numerical results of f′ for R = 5, 10, 50 and 107.